GCE: 502, Linear Algebra
May 2018

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Exercise 1:

Let $\mathbb{R}^{n\times n}$ be the set of n by n real matrices. Let V be the set of invertible matrices in $\mathbb{R}^{n\times n}$.

(i). Show that V is open in $\mathbb{R}^{n\times n}$.

(ii). Show that the operator A from V to $\mathbb{R}^{n\times n}$ defined by $A(V) = V^{-1}$ is continuous.

(iii). Show that V is dense in $\mathbb{R}^{n\times n}$.

Exercise 2:

Suppose that $M \in \mathrm{GL}_n(\mathbb{R})$ satisfies $M^k = I_n$ for some $k, n \in \mathbb{Z}$.

(i). Prove that $\det(M) \in \{\pm 1\}$.

(ii). Prove that the minimal polynomial of M divides $x^k - 1$. Does the same conclusion hold for the characteristic polynomial?

(iii). Let d_1 be the dimension of the eigenspace with eigenvalue 1. If k is odd, prove that $n - d_1$ is even.

(iv). Prove that condition (iii) is sufficient, that is, for any $n, d_1 \in \mathbb{N}$ satisfying

1. $n > 0$
2. $n - d_1 \geq 0$
3. $n - d_1 \equiv 0 \mod 2$

and for any odd k there exists $M \in \mathrm{GL}_n(\mathbb{R})$ with a d_1-dimensional eigenspace of eigenvalue 1, which satisfies $M^k = I_n$.

Exercise 3:

Compute the cofactor matrix of the $n \times n$ matrix A whose entries are -1 off diagonal and $n - 1$ on the main diagonal. What can you conclude about the invertibility of A?

Exercise 4:
Let V be a complex inner product space and let $\tau : V \to V$ be a linear operator on V with adjoint τ^*. Assume that subspaces $W_1, \ldots, W_k$ of V are all τ-invariant subspaces (i.e., that $\tau(W_i) \subseteq W_i$ for $i = 1, \ldots, k$). Recall the orthogonal complement of W_i is denoted by

$$W_i^\perp = \{v \in V \mid (\forall w \in W_i) \ \ (\langle v, w \rangle = 0)\}.$$

Prove that both $\sum_{i=1}^k W_i^\perp$ and $\cap_{i=1}^k W_i^\perp$ are τ^*-invariant.

Exercise 5:
Consider the space $\mathsf{Mat}_n(\mathbb{C})$ of $n \times n$ matrices with complex entries with Frobenius norm $\|A\| = \sqrt{\mathsf{tr}(AA^*)}$ where $*$ denotes the conjugate transpose map. Given $A \in \mathsf{Mat}_n(\mathbb{C})$, find a simple expression for the Hermitian matrix H which minimizes $\|A - H\|$.